For the Beauty of Ideas

Mathematics and

Imaginative Education

Pamela Hagen
pamelahagen@telus.net

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“All artists try for beauty of one kind or another,
mathematicians try for the beauty of ideas”

Donald Coxeter - 1999

The late Dr. H.S. MacDonald Coxeter could easily be said to be one of Canada’s greatest mathematicians, and it is Coxeter who has provided the inspiration for the title of this paper and the mathematical inspiration to M.C. Escher whose illustration Circle Limits III provides the graphical illustration.

This paper will take a look at the interplay of theories of imaginative education and a key area of mathematics, geometry. The theories that go on to form the imaginative framework of Kieran Egan will be discussed together with the particular theories of geometrical development of Dutch educator Pierre van Hiele. Having briefly looked at the theories of both Egan and van Hiele this paper will detail some practical examples of what the blending of these two theories has looked like, in terms of educational practice using knowledge and resources about Dr. Coxeter and his work in geometry. Finally a review of working with the students and the theories will conclude this written discussion.

It is hoped that some of the points raised in this paper will provide stimulus for more detailed research on the role of imagination in mathematics education and on the greater development of mathematical understanding for both teacher and learner. At the very least it is hoped that some discussion will be generated.

The two components of theory and practice are essential parts of a truly dialogical educational process. If theory is to have any relevance it is vital that its’ efficacy is examined. Likewise if practice is to have educational significance, particularly for the growth of the learner, then the basis from which the practice comes needs to be critically examined.

Of all Kieran Egan’s books The *Educated Mind* (2001) details most fully what both the theory and practice of imaginative education could look like when developed into general practice. The foundation for the imaginative framework is one that is a continuum of the immortal quest for knowledge and improvement. Egan sees this developing, from a philosopher's and theorist's point of view at least from the time of Plato, through Jean Jacques Rousseau, John Dewey and Jean Piaget to name but a few. As such Egan has developed a very emphatic context for the imaginative framework.

The incompatibilities within existing educational theories, and the impossibility of a truly child centred learning situation in current educational settings, is precisely *why* Egan proposes a reconstitution of the contemporary educational outlook. The starting points, or frame of reference, for this has to be the existing ideas. Egan believes that what must then be done is "…to let go of the old ideas and consider what sense of education is generated by taking "kinds of understanding" as the primary focus for thinking about education." (p.24). The child is now truly at *th*e centre, but it is a centre within society and culture, both of which the child will need to come to understand.

It is important then in Egan's view, that context and not content is what should direct educational practice in the ongoing cultural quest for knowledge. By looking at the context in which education takes place it is far easier to see the greater relevancy of Egan's "kinds of understanding" than it is to see a relevancy for the specific components of curriculum content. The importance of context is further supported by Kilpatrick et al. (2001) with particular reference to mathematical proficiency. Context is seen by both Egan and Kilpatrick et al., as a wide range of cultural, environmental and situational elements that can influence the instruction that takes place within education. How the elements of students, teacher and knowledge interact within any given context influences the development of cognitive growth in both a positive and negative manner.

Egan proposes a form of sequential structuring to the five "kinds of understanding" he feels make up the formation of an individuals understanding. A beginning attempt has been made to put the characteristics of these “levels” into a visual format in Table 1. The five progressive levels of understanding start with Somatic understanding, move on through Mythic understanding, roll rhythmically onwards through Romantic understanding, progress through Philosophic understanding and reflectively rest at Ironic understanding.

It is the three central kinds of understanding, Mythic, Romantic and Philosophic understanding that Egan concentrates on for reconstituting a contemporary view of education. For each of these kinds of understanding Egan has developed a detailed framework that include the characteristics of each kind of understanding. Binary opposites, and a particular emphasis of a story structure are represented in mythic understanding, transcendent qualities in Romantic understanding and the introduction of alternative approaches in Philosophic understanding.

While these kinds of understanding could be said to be in progressive stages, one kind of understanding definitely having to come before moving on to another, Egan very clearly points out that there are a number of characteristics that are not exclusive to only one kind of understanding. For example the use of pattern for visual development does not cease at the end of somatic understanding, the first level of understanding. Rather it continues to play a role in the development of subsequent kinds of understanding with a gradual lessening of influence as other characteristics are developing.

Egan feels there are specific cognitive tools used in the cultural quest for knowledge that are a central feature of the imaginative framework. The primary tool that is used and shows our rich cultural heritage, is that of language. It is this tool that will be used for the focus of this review.

Language is seen as more than the mode by which an individual expresses and receives their understanding of the world currently seen around them. Language, according to Egan, is also to be regarded as a means by which the mind is enlarged and develops a greater understanding of the world and all that it encompasses in thought, word and deed. It is in the latter use of language that is of primary concern to Egan.

In attempting to view education with a different pair of glasses, Egan is aided by and clearly adopts many of the theories of Lev Vygotsky. Building and adapting Vygotsky's ideas of intellectual development, the Eganistic view of growth in understanding takes a "tool" of intellectual development, i.e. language, and uses it to develop and grow the understanding of the individual *and* the individual as a member of the larger societal and cultural group. It is the growth of both of these kinds of understanding that will aid an individual's development through the five stages of development that Egan proposes.

Having briefly stated the central features of Egan's theory as they relate to epistemological practice it is pertinent to detail in a similar manner the theories of Pierre van Hiele.

Both Pierre van Hiele and his wife Dina van Hiele-Geldof worked as secondary mathematics educators in Holland in the 1960's and used their experiences to develop what are now known as the van Hiele levels of geometric development. Van Hiele (1986) does not want his work restricted to geometry or mathematics alone, rather it was to focus on levels of thinking.

> When I developed my levels approach it was aimed at the teaching and learning of geometry. This is an unnecessary restriction, however, the teaching and learning of other topics can be improved equally well with the same levels approach. (p.vii)

Originally numbered through five levels from level zero to level four, Table 2 below gives the details of the van Hiele levels which he further saw as "cognitive levels." (p.8) that exist around a hierarchical structure.

Starting with level zero the learner begins acquiring mathematical knowledge that includes specific terminology such as characteristics of geometric figures, e.g. numbers of sides, size of angles, etc. Gradually a learner begins to move through the levels but needs to pass through each of five phases that exist within each level before progressing to the next.

Fuys, Geddes and Tischler (1988) undertook a three-year study, which showed that of prime importance for a student's development through the levels, was the role of the teacher.

> "The interviewer/teacher plays an important role in helping students learn the subject and process and also in becoming aware of expectations and evaluating the quality of their own thinking." (p.74)

Van de Walle (2001) and Napitapulu (2001) further support the central role played by the teacher.

Important to the students movement from one level to another, van Hiele believes is the use of language. He feels that through the use of language to speak about and describe structures it is possible to find the links between given structures. Once the links are found it is possible to attain a higher level of thinking. Fuys et al. (1988) support the importance of appropriate language use to the development of geometric knowledge. In addition it is felt the use of manipulatives or concrete materials was a further supportive measure.

The use of language takes on an important additional role when discussed in the context of assessment. Mason (1997), Moran (1993) and Fuys et al. (1988) all point towards the written language of students providing extra understanding of how student are thinking for a teacher's assessment. Moran (1993) is particularly keen n the use of student writing for assessment,

> "…written language [it] provides access to how pupils think…Journal writing provided an insight into their thoughts and learning…By putting thoughts on paper the thinking process takes on a new dimension. (p.20)

Students for their own self-assessment and growth should also use written thought. Van de Walle provides an important clue to possible comparison between van Hiele and Egan, "The theory does not specify content or curriculum but rather can be applied to most activities." (p.311). This is because the van Hiele levels are oriented for the cognitive development of geometric understanding as the Egan set of understandings are for general cognitive growth.

What then does educational practice look like when dealing with these two theories of cognitive development, and where in the educational spectrum can they be applied? Although this is necessary for current purposes, Egan's framework could be applied to every level of education and learning. Its aim is to "…keep[ing] the educational energy alive…" (p.279). Egan passionately believes that they key to keeping the eternal quest for knowledge and learning going, and therefore some educational energy, is to stimulate the imagination.

The optimum time to begin this is at the beginning of the development of what Egan calls mythic understanding. It is within this domain that a child's imagination begins to develop. Not only would growth in understanding be stimulated from its' beginning but a path to continued growth and learning would be laid. Flexibility of intellect Egan believes would further aid in handling the "…complex demands of modern changing social conditions." (p.279).

The educational context in which mythic understanding lies is that of elementary education. Recent opportunities to study the use of Egan's mythic and romantic levels of understanding have provided impressive results in the growth of student's specific geometric cognitive development as well as general cognitive development. While these have not yet been incorporated into systematic theoretical studies, the examples most certainly provide suggestions for further research and study. The learning activities certainly embody many of the characteristics of both Egan's and van Hiele's levels of development.

Egan's key to cognitive development, as stated earlier, is the stimulation and use of the imagination. It is also important to remember that the context of use of both the mythic and romantic frameworks involves, according to Kilpatrick et al. (2001), the teacher as well as the student. While the learning focus is with the student, both student and teacher stand to gain in this type of approach.

It is in this context of mutual learning that I had the opportunity within the last school year to "test" the working of the Egan and van Hiele frameworks together. Opportunities to work with the van Hiele framework had occurred on two previous occasions. Working with Division One, a class of 28 mixed ability G5 students of Westwood Elementary school, meant that the students were predominantly within the Romantic Planning Framework. However it is important to note that characteristics of each level of understanding are not exclusive to only one level and that there can be a flow of characteristic qualities between levels, as with the visual development mentioned earlier.

Our topic was mathematics and our subject area was geometry. Artifacts and resources from my own graduate studies were used with the students. The transcendent qualities of life, energy and longevity were highlighted in the activities by the use of some personal artifacts. These were not necessary to working within the Egan frameworks but were selected because of the added opportunity to humanize the topic still further.

The first priceless artifact that was used was a very recent personal handwritten letter from one of Canada's greatest mathematician's, and perhaps one of the greatest ever geometry experts, Dr. H.S. MacDonald (Donald) Coxeter, see Appendix One. At the time of writing the letter in September 2002 Dr. Coxeter was 96 years of age and unfortunately was in declining health. I had the opportunity to meet Dr. Coxeter some two years earlier at a mathematics conference in Vancouver and had written to him asking about his views of the use of imagination in the learning of mathematics. His gracious reply and that of his daughter, written at the same time as the personal letter, provided the "spark" that began our study of geometry.

We were able to use readily available web based biographic details that augmented the information that we already had to begin building our narrative structure. During the course of our studies I decided to have the students complete an activity of building and then explaining a geometric paper construction. Both the students and their work can be seen in the figures below.

Figure 1 – Coxeter Stars

Figure 2 – Students of Division One & Coxeter Stars

A sample of the design was shown to the students and oral instructions were given as to how the design could be built. The level of engagement in the active phase of building the geometric designs was extremely high to the degree that few of the students wanted to take their regularly scheduled recess break! Their enjoyment was clearly noticeable in the comments made at the time and also can be clearly seen in the photograph of Figure Two.

The next stage of the activity required particular engagement of the students' imagination. They were asked to write a lesson plan for what they had done so that other students could also make the designs. During the first part students were asked to write explain what they had done in making their construction. Both words and diagrams were permitted in their writing. The concluding phase was to review the first written explanation and add as much mathematical detail as they could possibly think of to their original effort. Samples of student work are given in Appendix Two.

At the time of starting to make the geometric designs they did not have a name other than a generic descriptor of paper stars. The students and I collectively decided that we should now name the paper stars Coxeter Stars in honour of the mathematician who had written to us a few months earlier. The class was so proud of their work on the now named Coxeter Stars, that they asked the principal if they could put some around the school. This then blossomed into decorating the school gym for the parents evening as it was the Christmas season.

Collectively it was decided that we should send copies of the photographs to Dr. Coxeter and his daughter Susan Thomas with a Christmas note. This was done and shortly afterwards a further piece of Christmas communication was received from our photograph recipients, together with a copy of a Toronto magazine. The magazine had a full-length article and photographs about Dr. Coxeter, which provided further content and resources for our ongoing geometric studies. Other activities in our unit included examining the work of M.C. Escher whose original work of Circle Limits III hangs in the Coxeter/Thomas home. The students created manual tessellations during art lessons and used computer software, Tesslemania during lessons in the school computer lab.

Sadly a few months later, after we had completed the formal part of our studies, we received word that Dr. Coxeter had passed away at the end of March 2003, at age 96.

Clearly many characteristics of both the van Hiele and Egan theories were both used and demonstrated in the activities that took place with Division One of Westwood Elementary School. There was an obvious fascination from the students with both the life and work of Dr. Coxeter from the smallest detail about his own life as a schoolboy. In addition there was an overall sense of wonder as to how he could possibly explain mathematically the artistic work of his friend M.C. Escher.

Constructing the Coxeter Stars was clearly a van Hiele Level One activity that used the guided orientation, explicitation and free orientation phases. With regard to Egan's Romantic Planning Framework many characteristics were part of the activities, transcendent qualities, a sense of wonder and many opportunities for further investigation and discussion were raised to name just a few. Perhaps one particular aspect of Egan's Romantic framework, non-conformity, could be said to have been used by having the students write the instructions for future lessons that involved making Coxeter stars.

It was a moment of great pathos for the students to receive a copy of Dr. Coxeter's photographic obituary notice from his daughter, Susan Thomas that gave the two statements of "Life is not meant to be endured, it is to be celebrated" and "Don't waste time on what was; enjoy what is…" (appendix three).

The enjoyment of the students' and the degree of their engagement in all aspects of the activities are clear examples of what can be achieved with relative ease when using activities that follow the theories of Egan and van Hiele. The particular activities detailed above used my own personal artifacts. These are certainly nice to have for both the teacher and student, and can add a unique dimension, but they are not essential to being able to use either an imaginative framework for cognitive development or the theories of Egan and van Hiele.

At the present time most resources that are typically available in elementary classrooms and schools have been developed with a focus tuned to the development of knowledge and curriculum content and not human understanding. Therefore it can initially seem a challenge for individuals wanting to engage student's imaginations using Egan's framework. The Quest 2000 series of mathematics texts is such an example. Educational publishing companies in western Canada prepare their materials to meet provincial curriculum and the Western and Northern Canadian Protocol agreement between western and northern provinces in Canada.

However, there are some materials that can assist individuals with certain aspects of the framework approach such as *Mathematicians Are People Too*. A two volume series that has a collection of short biographies of famous mathematicians such as Archimedes, Euler and Pythagoras already in a narrative format. A companion set of three books, *Historical Connections in Mathematics, Vols. 1 – 3,* has prepared lesson materials suitable for intermediate and secondary students and teacher use. These can easily be adapted for younger students.

With the combination of text, web and multi-media resources abundance now available the "lack of resources" argument can only be used casually by those thinking about developing cognition rather and understanding.

It is somewhat more encouraging for those interested in teaching geometry more effectively, that the use of the van Hiele levels of geometric development are now appearing in mainstream educational resources (Van DeWalle, 2001). In addition, educational discussions around the subject of geometry and specifically the van Hiele levels are beginning to take place in both professional and publishing circles.

What is important, however, is to have the willingness on the part of the teacher to see learning framed from a cognitive development point of view rather than the more usual curriculum content. Unfortunately the manner in which the three principle imaginative frameworks of Egan are currently presented does not make them readily accessible by the majority of practicing teachers. This is not entirely surprising when one considers that most teachers, and others involved in education, have been trained in the traditional content and knowledge view of education. It is to assist with this matter that an attempt was made to prepare Table 1. It is presented here in its rudimentary form in a visual format that is hoped will invite critique, discussion and further development. If the concept of imaginative education can be made available in as many formats as possible to educators in all areas of education, then the opportunities to further disseminate the theory are potentially limitless.

A far richer notion of education has been put forward by Egan that can be worked into a contemporary educational contexts, as has been shown above, and also can clearly be seen by the samples of student work in Appendix 1. These samples of student work are discussed more fully in Hagen (2003). By blending the theories of Egan and van Hiele that have cognitive growth and learning as the prime foci, opportunities for the development of cognition in the young learner are available. By engaging the imaginative element not only is learning heightened of currently available knowledge but a path is laid that has the potential to help meet "…complex demands of modern changing social conditions." (Egan, 1997, p.279). The path can lead to a much brighter future for us all.

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"As for the role of imagination, I would say that all discovery requires imagination…"

Donald Coxeter
(personal communication, September 7, 2002)

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NB. The original of the front page illustration, Circle Limits III, hangs in the home of the late Prof. Coxeter in Toronto, Ontario. It was created after Escher had seen an illustration of Coxeter's Circle Limit I. Coxeter and Escher met in an artistic statement from very different perspectives. Escher came from a graphic art direction and Coxeter from a highly sophisticated mathematical bearing. The two became close friends as well as professional colleagues.

Table 2

Van Hiele's Levels of Thinking[Π]

Levels

Level 0
The student identifies, names and compares and operates on geometric figures (e.g. triangles, angles, intersecting or parallel lines) according to their appearance.

Level 1
The student analyses figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g. by folding, measuring, using a grid or diagram).

Level 2
The student logically interrelates previously discovered properties/rules by giving or following informal arguments.

Level 3
The student proves theorems deductively and establishes interrelationships among networks of theorems.

Level 4
The student establishes theorems in different postulational systems and analyses/compares these systems.

Phases within each level
(examples for level 0 – 1)

Information
The student gets acquainted with the working domain
(e.g. examines examples and non-examples).

Guided orientation:
The student does tasks involving different relations of the network that is to be formed (e.g. folding, measuring, looking for symmetry).

Explicitation: The student becomes conscious of the relations, tries to express them in words, and learns technical language, which accompanies the subject matter (e.g. expresses ideas about properties of figures).

Free orientation: The student learns, by doing more complex tasks, to find his/her own way in the network of relations (e.g. knowing properties of one kind of shapes, investigates these properties for a new shape, such as kites).

Integration: the student summarises all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available (e.g. properties of a figure are summarized).

[Π] Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. Reston VA: The National Council of Teachers of Mathematics.

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